Saving the divergence condition in CutFEM

with respect to the Darcy interface problem

Erik Nilsson (joint work with Sara Zahedi, Thomas Frachon, Peter Hansbo)

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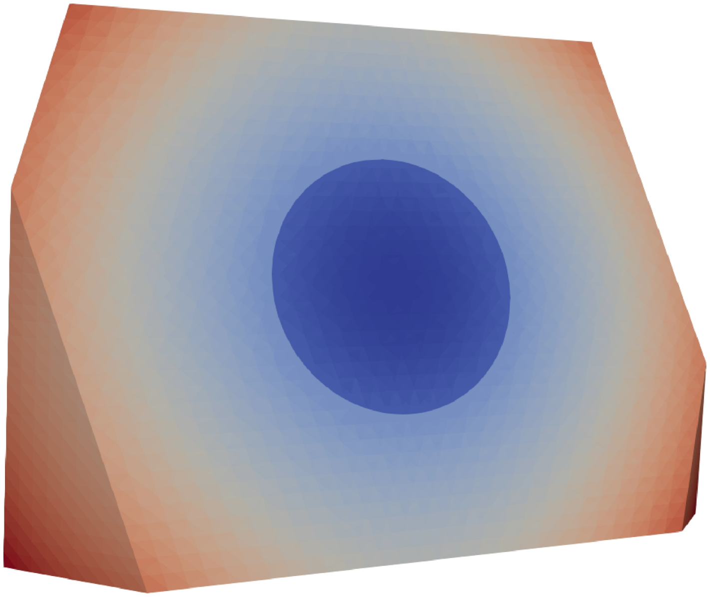

Interface problems as motivation for CutFEM

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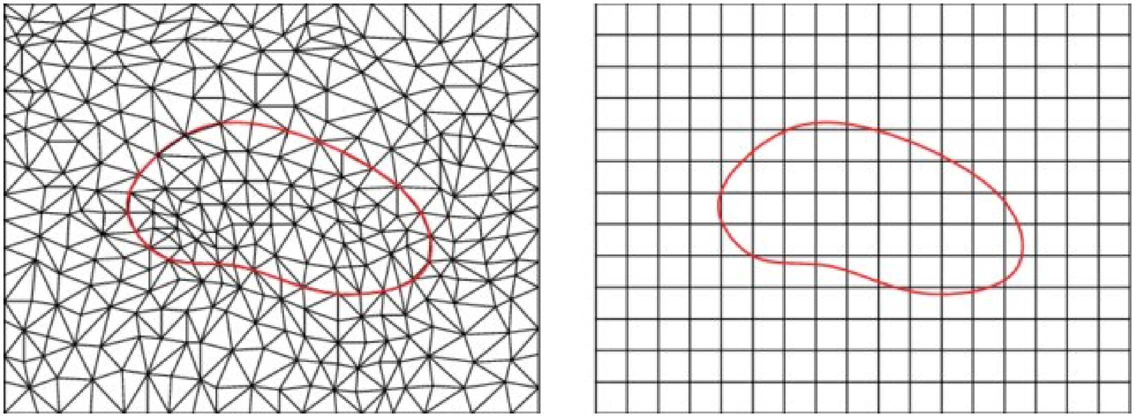

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[2]Residual-based a posteriori error estimation for immersed finite element methods. *Journal of Scientific Computing*, 81, 2019.

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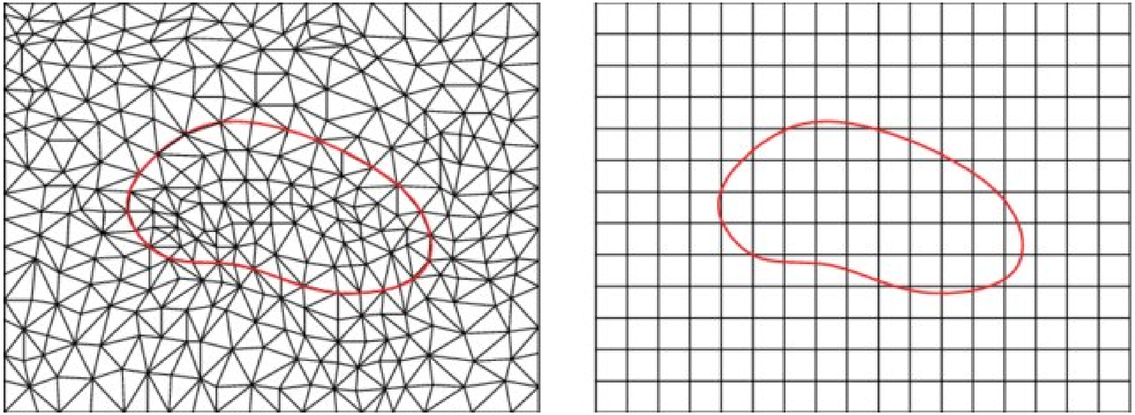

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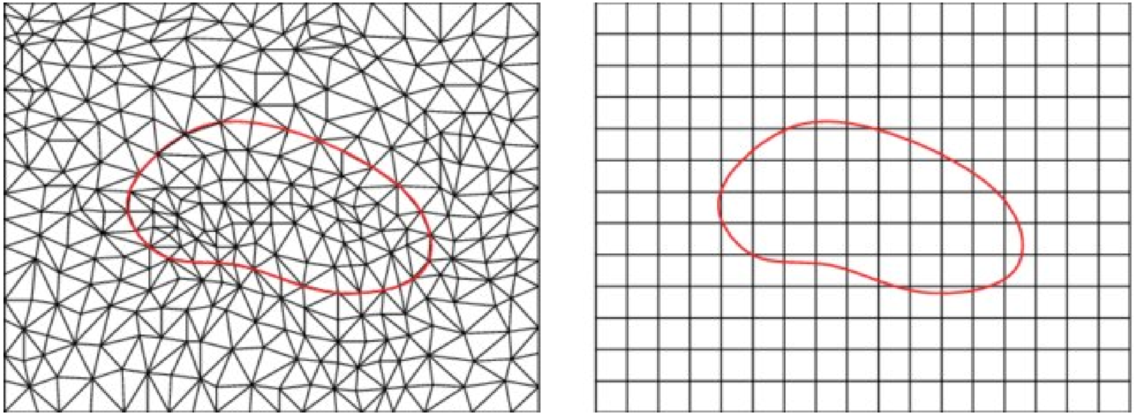
 [2]

CutFEM is an alternative to FEM where the mesh does not need to conform to the interface, and/or possibly also domain boundaries.

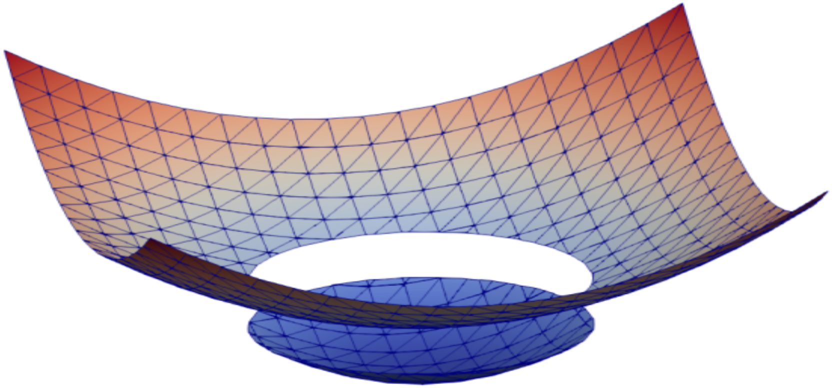
We call the mesh the *background mesh*.

[2]Residual-based a posteriori error estimation for immersed finite element methods. *Journal of Scientific Computing*, 81, 2019.

Consider the following Darcy interface problem. For data $\boldsymbol{\eta}, \boldsymbol{\eta_\Gamma}, \xi, \boldsymbol{f}, p_0, \hat{p}, g$

we look for solutions $\boldsymbol{u}, p$ to

$$
\begin{aligned}
\boldsymbol{\eta}\boldsymbol{u} - \nabla p &= \boldsymbol{f} && \text{in } \Omega \\
\mathrm{div}\boldsymbol{u} &= g && \text{in } \Omega \\
p &= p_0 && \text{on } \partial\Omega_p \\
\boldsymbol{u}\cdot\boldsymbol{n} &= u_0 && \text{on } \partial\Omega_u \\
[p] &= \boldsymbol{\eta_\Gamma}\{\boldsymbol{u}\cdot\boldsymbol{n}\} && \text{on } \Gamma \\
\{p\} &= \hat{p} + \xi\boldsymbol{\eta_\Gamma}[\boldsymbol{u}\cdot\boldsymbol{n}] && \text{on } \Gamma
\end{aligned}
$$



The Darcy interface problem weak formulation reads, for data $\boldsymbol{\eta}, \boldsymbol{\eta}_{\boldsymbol{\Gamma}}, \xi, \boldsymbol{f}, p_0, \hat{p}, g$:

The Darcy interface problem weak form

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Find $(\boldsymbol{u}, p) \in \boldsymbol{H}^{\text{div}}_{u_0, \partial\Omega_u}(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^{\text{div}}_{0, \partial\Omega_u} \\ b(\boldsymbol{u}, q) &= G(q) \quad \text{for all } q \in L^2(\Omega) \end{aligned}$$

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$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v}, p) = F(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^{\mathrm{div}}_{0,\partial\Omega_u} \tag{1}$$

$$b(\boldsymbol{u}, q) = G(q) \quad \text{for all } q \in L^2(\Omega) \tag{2}$$

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) = & \int_\Omega \boldsymbol{\eta}\boldsymbol{u}\cdot\boldsymbol{v} + \int_\Gamma \boldsymbol{\eta}_{\boldsymbol{\Gamma}}\{\boldsymbol{u}\cdot\boldsymbol{n}\}\{\boldsymbol{v}\cdot\boldsymbol{n}\} \\ & + \int_\Gamma \xi\boldsymbol{\eta}_{\boldsymbol{\Gamma}}[\boldsymbol{u}\cdot\boldsymbol{n}][\boldsymbol{v}\cdot\boldsymbol{n}] \\ b(\boldsymbol{v}, p) = & -\int_\Omega \mathrm{div}\,\boldsymbol{v}\, p \\ F(\boldsymbol{v}) = & \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - \int_{\partial\Omega_p} p_0\,\boldsymbol{v}\cdot\boldsymbol{n} - \int_\Gamma \hat{p}\,[\boldsymbol{v}\cdot\boldsymbol{n}] \\ G(q) = & -\int_\Omega g q \end{aligned}$$

[1]A mixed finite element method for Darcy flow in fractured porous media with non-matching grids. *ESAIM: Mathematical Modelling and Numerical Analysis*, 2012.

The differential operator for Darcy is $\mathrm{div}\colon \boldsymbol{H}^{\mathrm{div}}(\Omega) \to L^2(\Omega)$.

$\boldsymbol{H}^{\mathrm{div}}$-conforming FEM

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The finite element $\mathbf{RT}_k(\Omega) \subset \boldsymbol{H}^{\mathrm{div}}(\Omega)$ has the property $\mathrm{div}\,\mathbf{RT}_k(\Omega) \subset Q_k(\Omega)$, where $Q_k(\Omega)$ is the space of piecewise discontinuous Lagrange polynomials of order $k \geqslant 0$.

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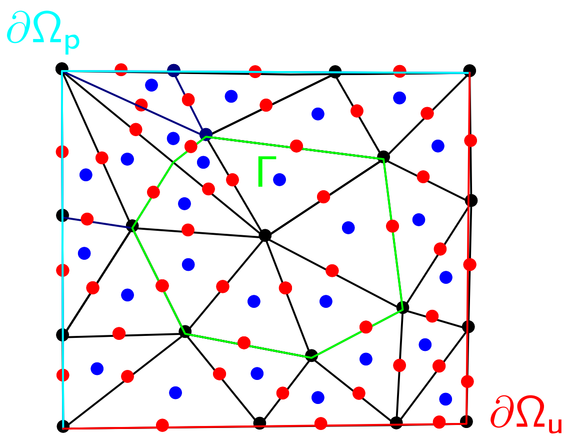

Red dots are $\mathbf{RT}_0$ degrees of freedom (DOFs)

Blue dots are Q_0 degrees of freedom (DOFs)

$\int_F \mathbf{w} \cdot \mathbf{n}$, $\mathbf{w} \in \mathbf{RT}$

$\int_K r$, $r \in Q$

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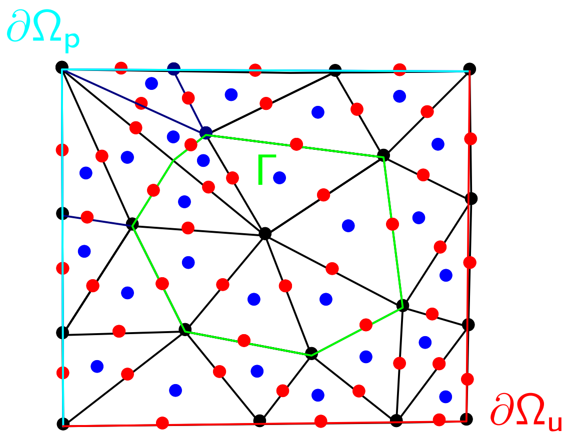

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$\int_F \mathbf{w}\cdot\mathbf{n},\ \mathbf{w}\in\mathbf{RT}$

$\int_K r,\ r\in Q$

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = F(\boldsymbol{v}_h)$$
$$b(\boldsymbol{u}_h, q_h) = G(q_h)$$

- $O(h^{k+1})$ convergence
- Pointwise exact divergence if $g \in Q_k(\Omega)$
- Well conditioned linear system

$$\longrightarrow A\hat{u} = b \Rightarrow \boxed{\hat{u} = A \setminus b}$$

CutFEM in a nutshell

From the background mesh Ω_h we define two active meshes $\Omega_{h,i}$ which have overlap at the interface. On each active mesh we construct a standard FEM space $V_{h,i}$.

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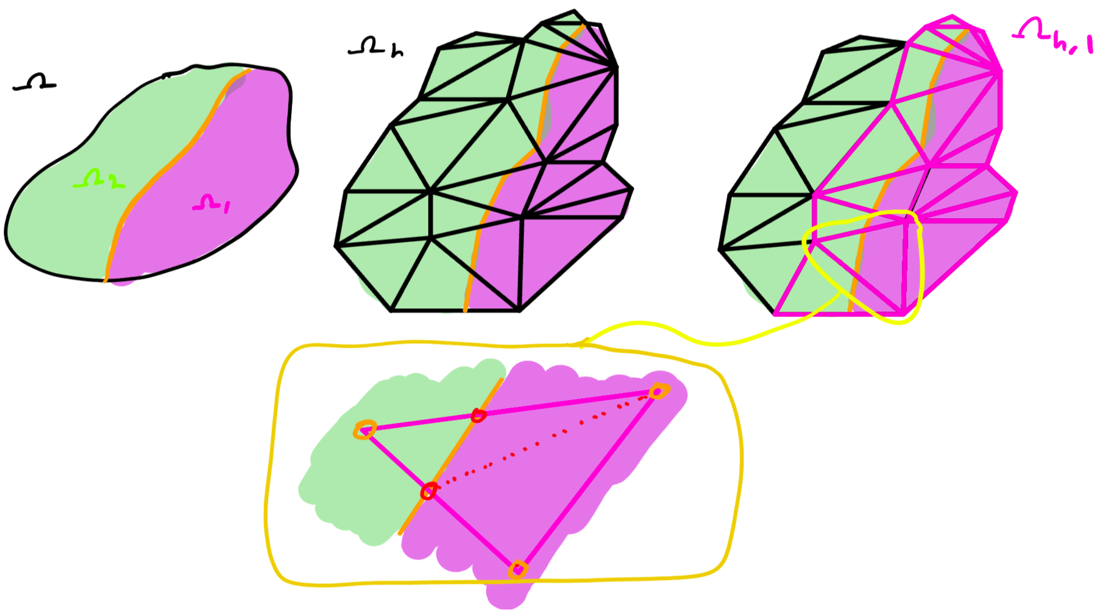

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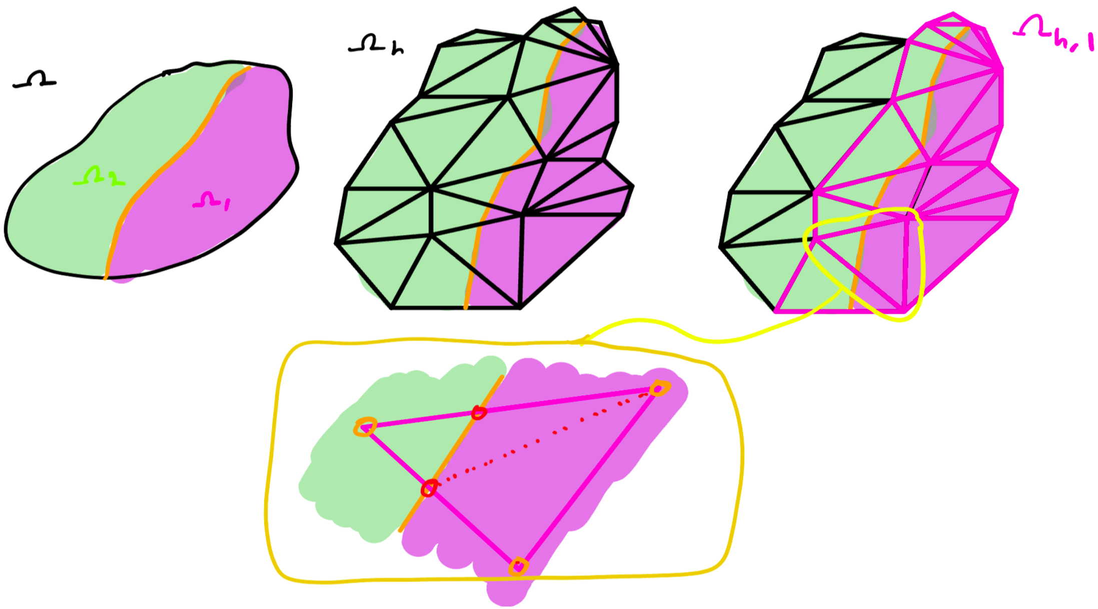
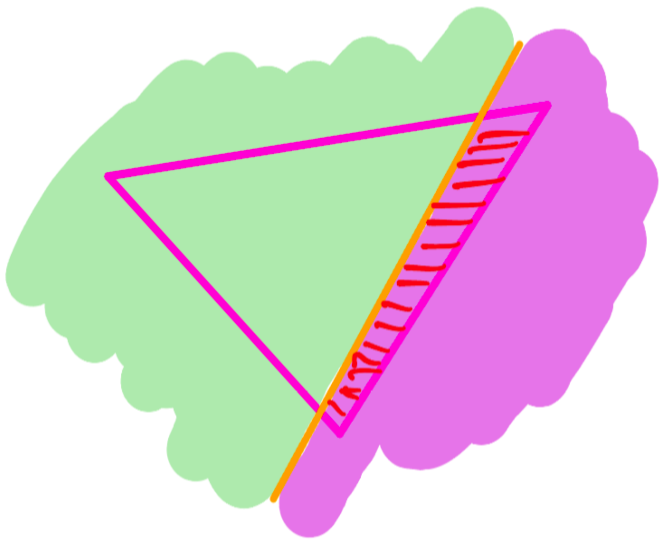


The CutFEM functions $u_h = (u_{h,1}, u_{h,2})$ are thus defined as elements of the product space $V_{h,1} \times V_{h,2}$, but each component is only ever integrated on its part of the physical domain Ω_i. The solution to a problem is then the $\mathbb{R}$-valued function

$$\hat{u}_h = \begin{cases} u_{h,1}, \text{ in } \Omega_1 \\ u_{h,2}, \text{ in } \Omega_2 \end{cases}$$

Unfitted elements K get intersected by the interface Γ, and the total support (of some) of the basis functions associated to K can get *arbitrarily* small.



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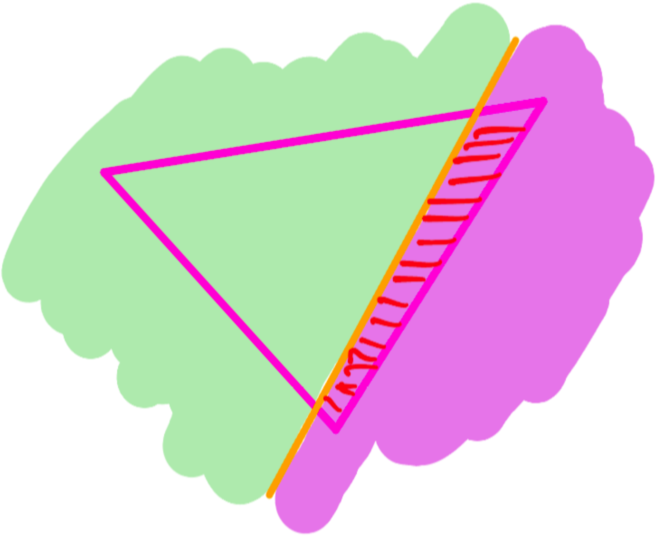

The resulting system matrix can accordingly be arbitrarily ill-conditioned.

Control of the condition number

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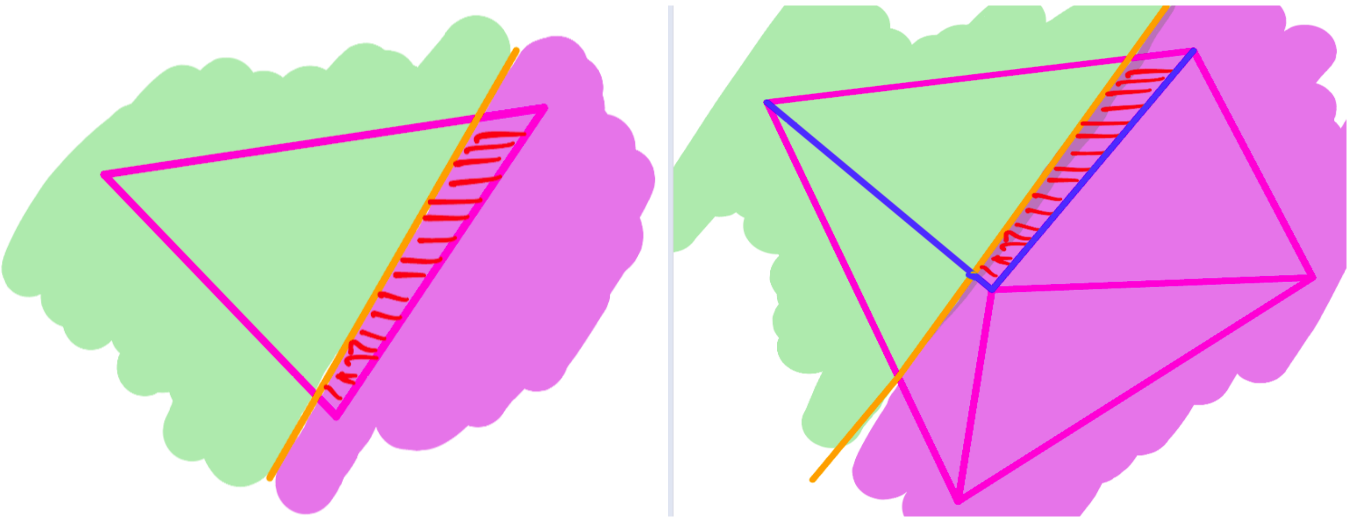

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$$
\begin{aligned}
s_a(\boldsymbol{w}, \boldsymbol{v}) &= \sum_{i \in \{1,2\}} \sum_{j=0}^{k+1} \sum_{F \in \mathfrak{F}_{h,i}} \tau_b h^{2j+1} \int_F [D^j \boldsymbol{w}][D^j \boldsymbol{v}] \\
s_b(\boldsymbol{w}, r) &= \sum_{i \in \{1,2\}} \sum_{j=0}^{k} \sum_{F \in \mathfrak{F}_{h,i}} \tau_b h^{2j+1} \int_F [D^j \operatorname{div} \boldsymbol{w}][D^j r]
\end{aligned}
$$

[4]A divergence preserving cut finite element method for Darcy flow. *ArXiv:2205.12023*, 2022.

CutFEM formulation of the Darcy interface problem

Let $\mathbf{RT}_k := \mathbf{RT}_k(\Omega_{h,1}) \times \mathbf{RT}_k(\Omega_{h,2}), Q_k := Q_k(\Omega_{h,1}) \times Q_k(\Omega_{h,2})$.

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Find $(\boldsymbol{u}_h, p_h) \in \mathbf{RT}_k \times Q_k$ such that

$$\begin{aligned} \mathcal{A}(\boldsymbol{u}_h, \boldsymbol{v}_h) + \mathcal{B}(\boldsymbol{v}_h, p_h) &= \mathcal{F}(\boldsymbol{v}_h) \quad \text{for all } \boldsymbol{v}_h \in \mathbf{RT}_k \\ \mathcal{B}_0(\boldsymbol{u}_h, q_h) &= G(q_h) \quad \text{for all } q_h \in Q_k \end{aligned}$$

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$$\begin{aligned} \mathcal{A}(\boldsymbol{w}, \boldsymbol{v}) &= a(\boldsymbol{w}, \boldsymbol{v}) + \int_{\partial\Omega_u} \lambda_u \boldsymbol{w} \cdot \boldsymbol{n}\, \boldsymbol{v} \cdot \boldsymbol{n} + s_a(\boldsymbol{w}, \boldsymbol{v}) \\ \mathcal{B}_0(\boldsymbol{w}, r) &= b(\boldsymbol{w}, r) - s_b(\boldsymbol{w}, r) \\ \mathcal{B}(\boldsymbol{w}, r) &= \mathcal{B}_0(\boldsymbol{w}, r) + \int_{\partial\Omega_u} \boldsymbol{w} \cdot \boldsymbol{n}\, r \\ \mathcal{F}(\boldsymbol{v}) &= F(\boldsymbol{v}) + \int_{\partial\Omega_u} \lambda_u u_0 \boldsymbol{v} \cdot \boldsymbol{n} \end{aligned}$$

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The terms s_a and s_b are ghost penalty stabilization terms used to control the stability of the method and the condition number of the resulting linear system.

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$$\mathcal{B}_0(\boldsymbol{u}_h, q_h) = G(q_h) \quad \text{for all } q_h \in Q_k \tag{4}$$

$$\begin{aligned} \mathcal{A}(\boldsymbol{w}, \boldsymbol{v}) &= a(\boldsymbol{w}, \boldsymbol{v}) + \int_{\partial\Omega_u} \lambda_u \boldsymbol{w} \cdot \boldsymbol{n}\, \boldsymbol{v} \cdot \boldsymbol{n} + s_a(\boldsymbol{w}, \boldsymbol{v}) \\ \mathcal{B}_0(\boldsymbol{w}, r) &= b(\boldsymbol{w}, r) - s_b(\boldsymbol{w}, r) \\ \mathcal{B}(\boldsymbol{w}, r) &= \mathcal{B}_0(\boldsymbol{w}, r) + \int_{\partial\Omega_u} \boldsymbol{w} \cdot \boldsymbol{n}\, r \\ \mathcal{F}(\boldsymbol{v}) &= F(\boldsymbol{v}) + \int_{\partial\Omega_u} \lambda_u u_0 \boldsymbol{v} \cdot \boldsymbol{n} \end{aligned}$$

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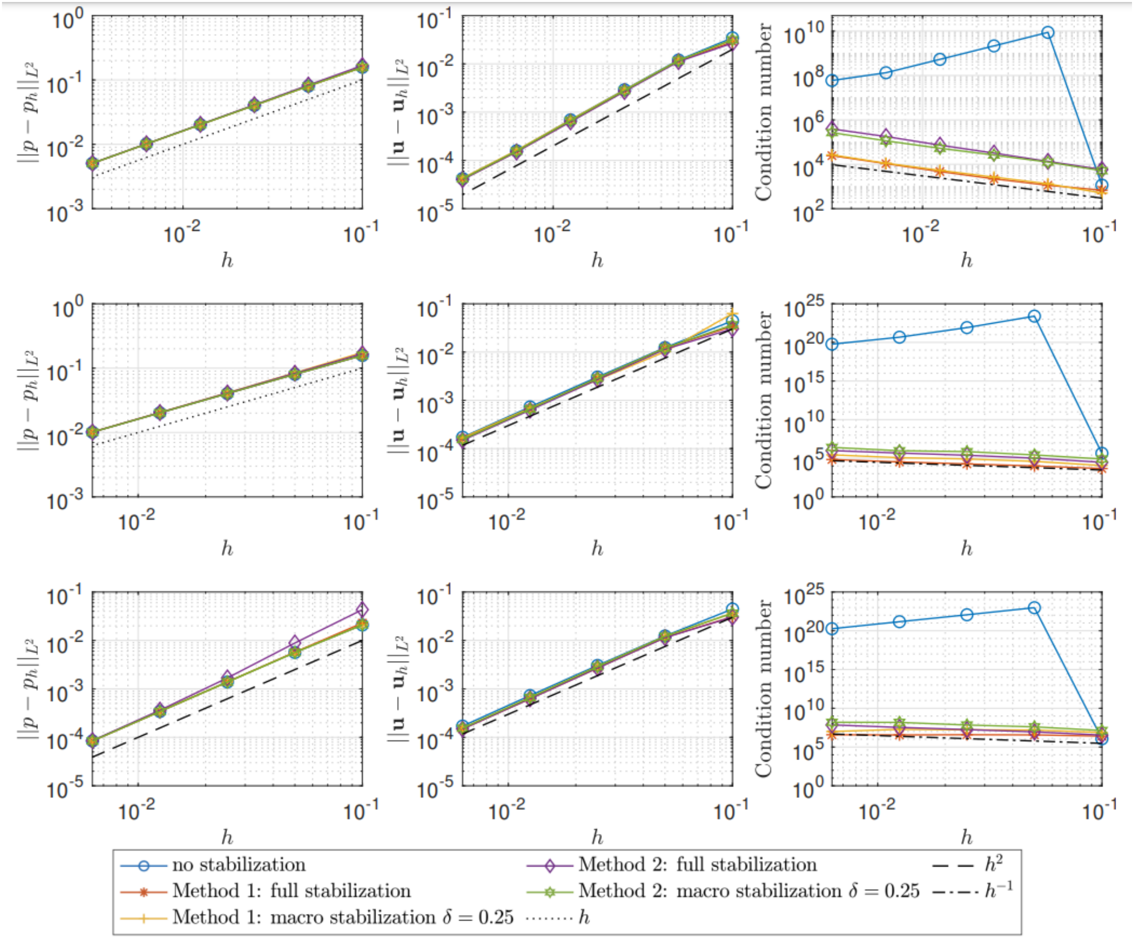
Further, s_b is designed so that also the divergence condition is satisfied.

Numerical results: $\partial\Omega = \partial\Omega_p$

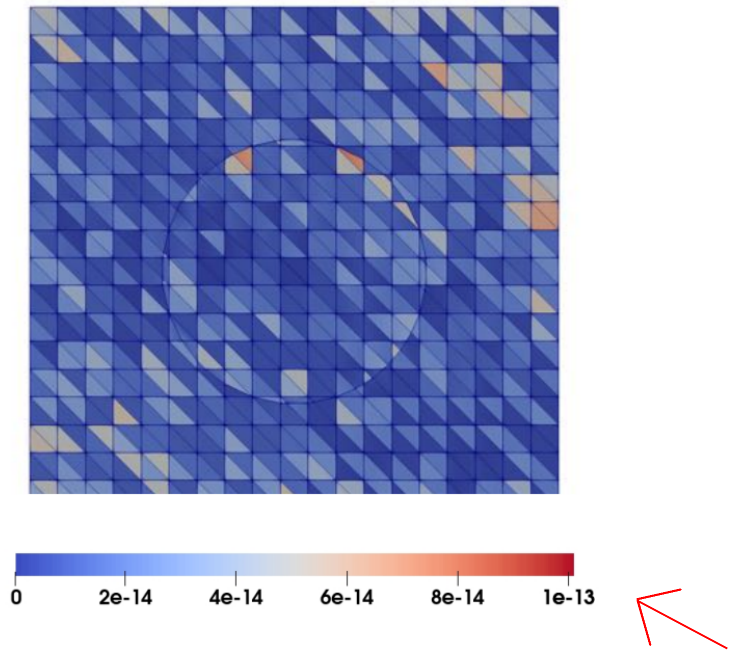
Example 1: interface problem, boundary conditions (BC) $\partial\Omega = \partial\Omega_p$, divergence inside Q_0



[4]A divergence preserving cut finite element method for Darcy flow. *ArXiv:2205.12023*, 2022.

Numerical results: divergence condition

Example 1: divergence is in Q_0 so we expect an error of order machine precision



Plotted above is $\|\operatorname{div}\boldsymbol{u}_h - g\|_\infty$.

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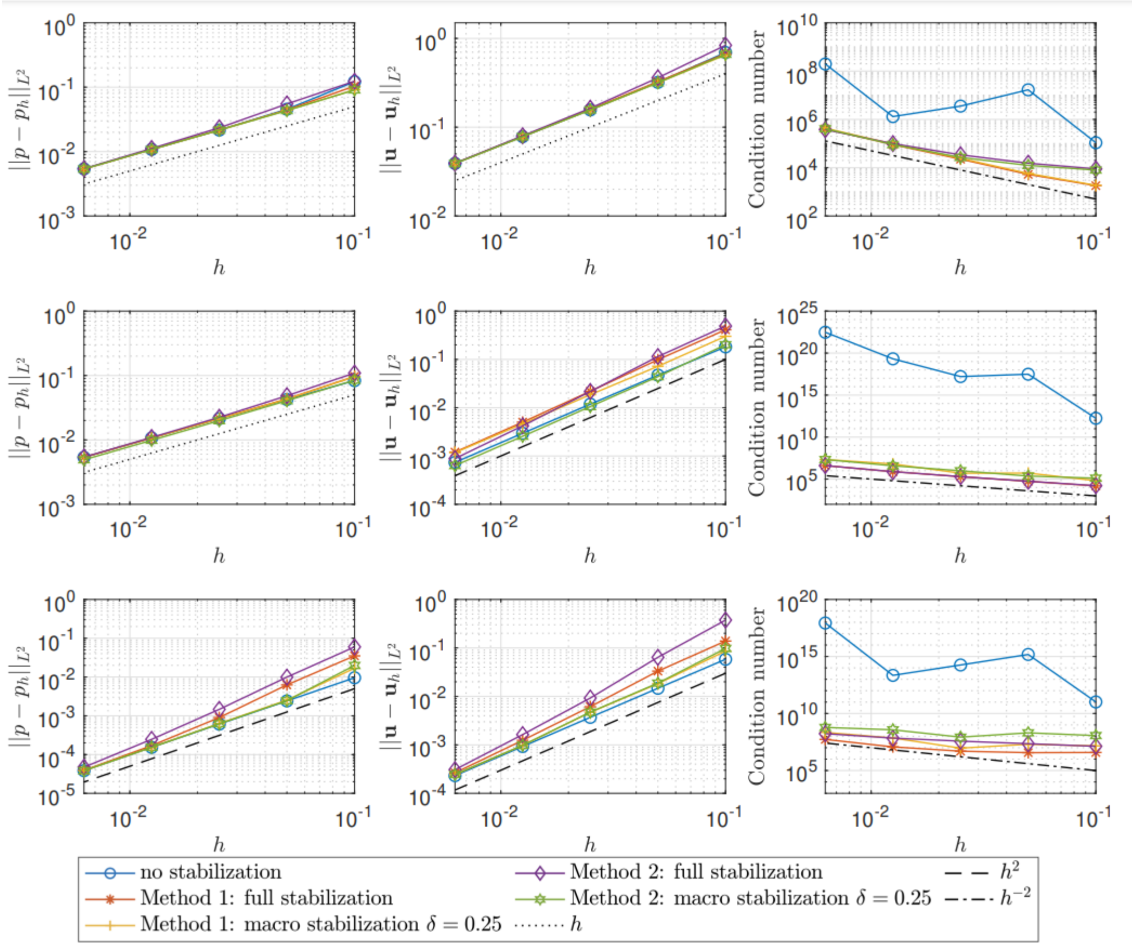
Example 2: fictitious domain (no interface) $\Gamma = \emptyset$, BC $\partial\Omega = \partial\Omega_u$, divergence outside Q_k



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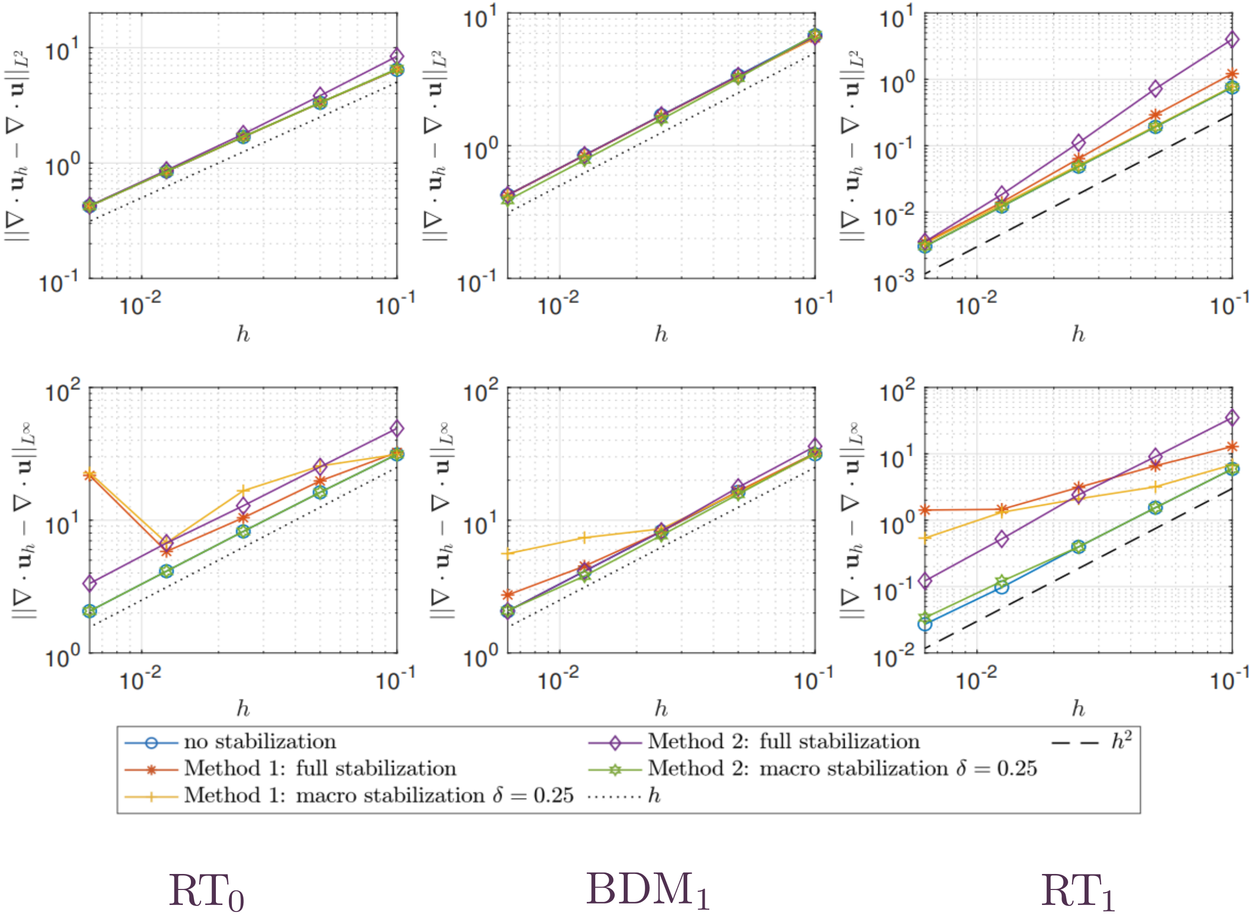
Numerical results 2: divergence condition

Example 2: divergence is not in Q_k for any $k \geq 0$ so the best we can hope for is optimal convergence order $k+1$ in $\|\cdot\|_\infty$.



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Why does it work?

Theorem 1. *Let $g \in Q_k(\Omega) \cap C^k(\mathfrak{F}_{h,1}) \cap C^k(\mathfrak{F}_{h,2})$, the space of piecewise discontinuous polynomials which have k continuous derivatives over the stabilization faces. Then the CutFEM solution $\boldsymbol{u}_h \in \mathbf{RT}_k$ to (6)+(7) satisfies*

$$div\, \boldsymbol{u}_h = g$$

Proof. Consider (7): for all $q_h \in Q_k$

$$\int_\Omega \operatorname{div} \boldsymbol{u}_h\, q_h + \sum_{i\in\{1,2\}} \sum_{j=0}^{k} \sum_{F\in\mathfrak{F}_{h,i}} \tau_b h^{2j+1} \int_F [D_{\boldsymbol{n}}^j \operatorname{div} \boldsymbol{u}_h][D_{\boldsymbol{n}}^j q_h] - \int_\Omega g\, q_h = 0.$$

Since $\operatorname{div} \mathbf{RT}_k \subset Q_k$ we can choose $q_h = \operatorname{div} \boldsymbol{u}_h - g$. Then,

$$0 = \int_\Omega (\operatorname{div} \boldsymbol{u}_h - g)^2 + \sum_{i\in\{1,2\}} \sum_{j=0}^{k} \sum_{F\in\mathfrak{F}_{h,i}} \tau_b h^{2j+1} \int_F [D_{\boldsymbol{n}}^j (\operatorname{div} \boldsymbol{u}_h - g)]^2 \quad \Leftrightarrow$$

$$0 = \|\operatorname{div} \boldsymbol{u}_h - g\|_\Omega^2 + s_p(\operatorname{div} \boldsymbol{u}_h - g, \operatorname{div} \boldsymbol{u}_h - g) \gtrsim \|\operatorname{div} \boldsymbol{u}_h - g\|_{\Omega_h}^2 \geqslant 0$$

Consequently $\operatorname{div} \boldsymbol{u}_h = g$ since $\|\cdot\|_{\Omega_h}$ is a norm for Q_k. □

Previous research [1][3]	$k=$ 0	1	2	...
Optimal convergence order	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Controlled condition number	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Satisfies divergence condition	$\checkmark$	X	X	X

CutFEM with new saddle-point stabilization [4]	$k=$ 0	1	2	...
Optimal convergence order	$\checkmark$	$\checkmark$	$\checkmark$	$(\checkmark)$
Controlled condition number	$\checkmark$	$\checkmark$	$\checkmark$	$(\checkmark)$
Satisfies divergence condition	$\checkmark$	$\checkmark$	$\checkmark$	$(\checkmark)$

Performance summary & future work

Previous research [1][3]	$k=$ 0	1	2	...
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Controlled condition number	$\checkmark$	$\checkmark$	$\checkmark$	$(\checkmark)$
Satisfies divergence condition	$\checkmark$	$\checkmark$	$\checkmark$	$(\checkmark)$

For the future, possible directions:

-The method can be extended to Stokes equations (we are currently working on this.)

-Well-posedness proofs (we have almost completed this.)

–Navier-Stokes equations, moving interfaces

–Benchmark studies, comparison with other methods etc.

–Adapting the method to coupled problems which are not toy problems

Questions?

Bibliography

[1] A mixed finite element method for Darcy flow in fractured porous media with non-matching grids. *ESAIM: Mathematical Modelling and Numerical Analysis*, 2012.

[2] Residual-based a posteriori error estimation for immersed finite element methods. *Journal of Scientific Computing*, 81, 2019.

[3] A cut finite element method for the Darcy problem. *ArXiv:2111.09922*, 2021.

[4] A divergence preserving cut finite element method for Darcy flow. *ArXiv:2205.12023*, 2022.